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LETTER TO THE EDITOR

Multivariable continuous Hahn and Wilson polynomials related to integrable difference systems

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Abstract. Multivariable generalizations of the continuous Hahn and Wilson polynomials are introduced as eigenfunctions of rational Ruijsenaars-type difference systems with an external field.

In a by now famous pioneering paper, Calogero studied (essentially) the spectrum and eigenfunctions of a quantum system of N one-dimensional particles placed in a harmonic well and interacting by means of an inverse square pair potential [1]. The spectrum of the system, which is discrete due to the harmonic confinement, turns out to be remarkably simple: it coincides with that of *non-interacting* particles in a harmonic well, up to an overall shift of the energy. The structure of the corresponding eigenfunctions is also quite simple: they are the product of a factorized (Jastrow-type) ground-state wavefunction and certain symmetric polynomials. Recently, a rather explicit construction of these polynomials was given in terms of raising and lowering operators [2].

Some time after its introduction, Olshanetsky and Perelomov realized that the Calogero system can be naturally generalized within a Lie-theoretic setting, such that for each (normalized) root system there exists an associated Calogero-type quantum model [3]. From this viewpoint, the original model corresponds to the root system A_{N-1}. If one restricts attention to classical (i.e. non-exceptional) normalized root systems, then the Hamiltonians of the corresponding Calogero models split into two types.

Type A: root system A_{N-1}

$$H_{\mathrm{Cal,A}} = -\sum_{1\leqslant j\leqslant N} \partial_j^2 + g(g-1)\sum_{1\leqslant j\neq k\leqslant N}(x_j-x_k)^{-2} + \omega_0^2\sum_{1\leqslant j\leqslant N} x_j^2 \qquad (1)$$

Type B: root systems B_N and D_N ($g_0=0$)

$$H_{\mathrm{Cal,B}} = -\sum_{1\leqslant j\leqslant N} \partial_j^2 + g(g-1)\sum_{1\leqslant j\neq k\leqslant N}((x_j-x_k)^{-2}+(x_j+x_k)^{-2}) + \sum_{1\leqslant j\leqslant N}(g_0(g_0-1)x_j^{-2}+\omega_0^2x_j^2) \qquad (2)$$

($\partial_j=\partial/\partial x_j$). For the very special case $N=1$, (the polynomial parts of) the eigenfunctions of these Hamiltonians amount to Hermite polynomials (type A) and Laguerre polynomials (type B). If the external harmonic field is switched off, i.e. for $\omega_0=0$, then the spectrum of the Hamiltonian becomes continuous (assuming $g, g_0\geqslant 0$), and the eigenfunctions are no longer polynomials but give rise to multivariable families of Bessel-type functions [4].

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Several years ago, Ruijsenaars introduced a relativistic generalization of the type A model without an external field [5]. The Hamiltonian of this relativistic system is given by an analytic difference operator that becomes $H_{\mathrm{Cal,A}}$ (1) (with $\omega_0 = 0$) after sending the step size to zero. (This transition may be interpreted as the non-relativistic limit.) More recently we found similar difference versions for the type A model with $\omega_0 \neq 0$ and for the type B model [6]. In this letter we study the eigenfunctions of these two difference models. For the type A difference version this will lead to a multivariable generalization of the continuous Hahn polynomials [7, 8], whereas for type B we will find a multivariable version of the Wilson polynomials [9, 10]. By sending the step size to zero, we recover the spectrum and eigenfunctions of the Calogero Hamiltonians $H_{\mathrm{Cal,A}}$ (1) and $H_{\mathrm{Cal,B}}$ (2).

Type A: continuous Hahn case. The Hamiltonian of the (rational) difference Ruijsenaars system with external field is given by the second-order difference operator [6]

$$
\begin{aligned}
H_{\mathrm{A}} = \sum_{1\leqslant j\leqslant N} \Bigg(& w_+^{1/2}(x_j) \prod_{k\neq j} v^{1/2}(x_j - x_k) \mathrm{e}^{-\mathrm{i}\partial_j} \prod_{k\neq j} v^{1/2}(x_k - x_j) w_-^{1/2}(x_j) \\
& + w_-^{1/2}(x_j) \prod_{k\neq j} v^{1/2}(x_k - x_j) \mathrm{e}^{\mathrm{i}\partial_j} \prod_{k\neq j} v^{1/2}(x_j - x_k) w_+^{1/2}(x_j) \\
& - w_+(x_j) \prod_{k\neq j} v(x_j - x_k) - w_-(x_j) \prod_{k\neq j} v(x_k - x_j) \Bigg)
\end{aligned}
\tag{3}
$$

where $(\mathrm{e}^{\pm\mathrm{i}\partial_j}\Psi)(x_1, \ldots, , x_N) = \Psi(x_1, \ldots, x_{j-1}, x_j \pm i, x_{j+1}, \ldots, x_N)$ and

$$
v(z) = 1 + g/(\mathrm{i}z) \qquad w_+(z) = (a_+ + \mathrm{i}z)(b_+ + \mathrm{i}z) \qquad w_-(z) = (a_- - \mathrm{i}z)(b_- - \mathrm{i}z).
\tag{4}
$$

We will assume

$$
g \geqslant 0 \qquad a_- = \overline{a}_+ \qquad b_- = \overline{b}_+ \qquad \mathrm{Re}(a_+, a_-, b_+, b_-) > 0
\tag{5}
$$

which ensures, in particular, that the Hamiltonian is formally self-adjoint. In order to solve the eigenvalue problem for H_{A} (3)–(5) in the Hilbert space of square integrable permutation-invariant functions, we introduce the weight function

$$
\Delta_{\mathrm{A}} = \prod_{1\leqslant j\neq k\leqslant N} \frac{\Gamma(g + \mathrm{i}(x_j - x_k))}{\Gamma(\mathrm{i}(x_j - x_k))} \prod_{1\leqslant j\leqslant N} \Gamma(a_+ + \mathrm{i}x_j)\Gamma(b_+ + \mathrm{i}x_j)\Gamma(a_- - \mathrm{i}x_j)\Gamma(b_- - \mathrm{i}x_j).
\tag{6}
$$

Condition (5) implies that Δ_{A} is positive. Since the transformed operator

$$
\begin{aligned}
\mathcal{H}_{\mathrm{A}} &= \Delta_{\mathrm{A}}^{-1/2} H_{\mathrm{A}} \Delta_{\mathrm{A}}^{1/2} \\
&= \sum_{1\leqslant j\leqslant N} \Bigg(w_+(x_j) \prod_{k\neq j} v(x_j - x_k)(\mathrm{e}^{-\mathrm{i}\partial_j} - 1) + w_-(x_j) \prod_{k\neq j} v(x_k - x_j)(\mathrm{e}^{\mathrm{i}\partial_j} - 1) \Bigg)
\end{aligned}
\tag{7}
$$

clearly annihilates constant functions, it follows that $\Delta_{\mathrm{A}}^{1/2}$ is an eigenfunction of H_{A} with eigenvalue zero. This eigenfunction corresponds to the ground state. The excited states are a product of $\Delta_{\mathrm{A}}^{1/2}$ and symmetric polynomials associated with the weight function Δ_{A} (6).

Specifically, one has

$$
H_{\mathrm{A}}\Psi_n = E_{\mathrm{A}}(n)\Psi_n \qquad n \in \mathbb{Z}^N,\ n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0
\tag{8}
$$

with eigenvalues reading

$$
E_{\mathrm{A}}(n) = \sum_{1\leqslant j\leqslant N} n_j(n_j + a_+ + a_- + b_+ + b_- - 1 + 2(N - j)g)
\tag{9}
$$

and eigenfunctions of the form

$$\Psi_n(x) = \Delta_{\mathrm{A}}^{1/2} p_{n,\mathrm{A}}(x) \tag{10}$$

where $p_{n,\mathrm{A}}(x)$ denotes the symmetric polynomial determined by the conditions

$$\text{A1} \quad p_{n,\mathrm{A}}(x) = m_n(x) + \sum_{n'<n} c_{n,n'} m_{n'}(x) \qquad c_{n,n'} \in \mathbb{C}$$

$$\text{A2} \quad \int_{\mathbb{R}^N} p_{n,\mathrm{A}}(x) \overline{m_{n'}(x)} \Delta_{\mathrm{A}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_N = 0 \qquad \text{if } n' < n.$$

Here, the functions $m_n(x)$ denote the basis of monomial symmetric functions

$$m_n(x) = \sum_{n' \in S_N(n)} x_1^{n'_1} \cdots x_N^{n'_N} \qquad n \in \mathbb{Z}^N,\ n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0 \tag{11}$$

and the partial order of the basis elements is defined by

$$n' \leqslant n \qquad \text{iff} \qquad \sum_{1 \leqslant j \leqslant k} n'_j \leqslant \sum_{1 \leqslant j \leqslant k} n_j \qquad \text{for } k = 1, \ldots, N \tag{12}$$

($n' < n$ if $n' \leqslant n$ and $n' \neq n$).

The proof of the above statement amounts to showing that the polynomials $p_{n,\mathrm{A}}(x)$ are eigenfunctions of the transformed operator $\mathcal{H}_{\mathrm{A}}$ (7). This follows from the fact that $\mathcal{H}_{\mathrm{A}}$ is both triangular with respect to the monomial basis

$$(\mathcal{H}_{\mathrm{A}} m_n)(x) = \sum_{n' \leqslant n} [\mathcal{H}_{\mathrm{A}}]_{n,n'} m_{n'}(x) \tag{13}$$

and symmetric with respect to the L^2 inner product with weight function Δ_{A} (6). The eigenvalues $E_{\mathrm{A}}(n)$ (9) are obtained by computing the diagonal matrix elements $[\mathcal{H}_{\mathrm{A}}]_{n,n}$, i.e. the leading coefficients in expansion (13) of $(\mathcal{H}_{\mathrm{A}} m_n)(x)$ in monomial symmetric functions $m_{n'}(x)$.

By definition, the polynomial $p_{n,\mathrm{A}}(x)$ amounts to $m_n(x)$ minus its orthogonal projection in $L^2(\mathbb{R}^N, \Delta_{\mathrm{A}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_N)$ onto $\mathrm{span}\{m_{n'}\}_{n'<n}$. For $N = 1$ these polynomials reduce to (monic) continuous Hahn polynomials [7, 8].

Type B: Wilson case. Multivariable Wilson polynomials are obtained in much the same manner as their continuous Hahn counterparts, except that in addition to being permutation symmetric now everything also becomes even in x_j, $j = 1, \ldots, N$. The Hamiltonian of the type B version of the difference Ruijsenaars system reads [6]

$$H_{\mathrm{B}} = \sum_{1 \leqslant j \leqslant n} \left(V_{+j}^{1/2} \mathrm{e}^{-\mathrm{i}\partial_j} V_{-j}^{1/2} + V_{-j}^{1/2} \mathrm{e}^{\mathrm{i}\partial_j} V_{+j}^{1/2} - V_{+j} - V_{-j} \right) \tag{14}$$

with

$$V_{\pm j} = w(\pm x_j) \prod_{k \neq j} v(\pm x_j + x_k) v(\pm x_j - x_k) \tag{15}$$

$$v(z) = 1 + g/(\mathrm{i}z) \qquad w(z) = \frac{(a + \mathrm{i}z)(b + \mathrm{i}z)(c + \mathrm{i}z)(d + \mathrm{i}z)}{2\mathrm{i}z(2\mathrm{i}z + 1)} \tag{16}$$

and

$$g \geqslant 0 \qquad a, b, c, d > 0. \tag{17}$$

One now has

$$H_{\mathrm{B}} \Psi_n = E_{\mathrm{B}}(n) \Psi_n \qquad n \in \mathbb{Z}^N,\ n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0 \tag{18}$$

with

$$E_{\mathrm{B}}(n) = \sum_{1\leqslant j\leqslant N} n_j(n_j + a + b + c + d - 1 + 2(N-j)g) \qquad \Psi_n(x) = \Delta_{\mathrm{B}}^{1/2} p_{n,\mathrm{B}}(x^2) \tag{19}$$

where

$$\Delta_{\mathrm{B}} = \prod_{1\leqslant j\neq k\leqslant N} \left| \frac{\Gamma(g+\mathrm{i}(x_j+x_k))\Gamma(g+\mathrm{i}(x_j-x_k))}{\Gamma(\mathrm{i}(x_j+x_k))\Gamma(\mathrm{i}(x_j-x_k))} \right| \times \prod_{1\leqslant j\leqslant N} \left| \frac{\Gamma(a+\mathrm{i}x_j)\Gamma(b+\mathrm{i}x_j)\Gamma(c+\mathrm{i}x_j)\Gamma(d+\mathrm{i}x_j)}{\Gamma(2\mathrm{i}x_j)} \right|^2 \tag{20}$$

and $p_{n,\mathrm{B}}(x^2)$ is the even symmetric polynomial determined by the conditions

B1 $\quad p_{n,\mathrm{B}}(x^2) = m_n(x^2) + \sum_{n'<n} c_{n,n'} m_{n'}(x^2) \qquad c_{n,n'} \in \mathbb{C}$

B2 $\quad \int_{\mathbb{R}^N} p_{n,\mathrm{B}}(x^2)\overline{m_{n'}(x^2)}\Delta_{\mathrm{B}}\,\mathrm{d}x_1\cdots\mathrm{d}x_N = 0 \qquad$ if $n' < n$.

Here the functions $m_n(x^2)$ stand for the basis of even symmetric monomials

$$m_n(x^2) = \sum_{n'\in S_N(n)} x_1^{2n'_1}\cdots x_N^{2n'_N} \qquad n\in\mathbb{Z}^N,\ n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0 \tag{21}$$

and the partial order of the basis elements is the same as before (see (12)).

The polynomials $p_{n,\mathrm{B}}(x^2)$ are, of course, again eigenfunctions of the transformed operator

$$\mathcal{H}_{\mathrm{B}} = \Delta_{\mathrm{B}}^{-1/2} H_{\mathrm{B}} \Delta_{\mathrm{B}}^{1/2} = \sum_{1\leqslant j\leqslant N} \left(w(x_j)\prod_{k\neq j} v(x_j+x_k)v(x_j-x_k)(\mathrm{e}^{-\mathrm{i}\partial_j}-1) + w(-x_j)\prod_{k\neq j} v(-x_j+x_k)v(-x_j-x_k)(\mathrm{e}^{\mathrm{i}\partial_j}-1)\right). \tag{22}$$

For $N=1$ they reduce to (monic) Wilson polynomials [9, 10]

Transition to the Calogero system. If we substitute $x_j \to \beta^{-1}x_j$ (so $\partial_j \to \beta\partial_j$) and $a_+, a_- \to (\beta^2\omega_0)^{-1}$, $b_+, b_- \to (\beta^2\omega_0')^{-1}$ in our type A difference Hamiltonian and multiply by $\beta^2\omega_0\omega_0'$, then we arrive at a Hamiltonian of the form H_{A} (3) with $\exp(\pm\mathrm{i}\partial_j)$ replaced by $\exp(\pm\mathrm{i}\beta\partial_j)$ and

$$v(z) = 1 + \beta g/(\mathrm{i}z) \qquad w_\pm(z) = \beta^{-2}(1\pm\mathrm{i}\beta\omega_0 z)(1\pm\mathrm{i}\beta\omega_0' z). \tag{23}$$

By sending the step size β to zero the difference operator becomes a differential operator of the form $H_{\mathrm{Cal,A}} - \varepsilon_{0,\mathrm{A}}$, where $H_{\mathrm{Cal,A}}$ is given by (1) with ω_0 replaced by $\omega_0+\omega_0'$, and $\varepsilon_{0,\mathrm{A}}$ is a constant with value

$$\varepsilon_{0,\mathrm{A}} = (\omega_0+\omega_0')N(1+(N-1)g). \tag{24}$$

In this limit the weight function Δ_{A} (6) becomes (after dividing by a divergent numerical factor)

$$\Delta_{\mathrm{Cal,A}} = \prod_{1\leqslant j\neq k\leqslant N} |x_j-x_k|^g \prod_{1\leqslant j\leqslant N} \mathrm{e}^{-(\omega_0+\omega_0')x^2}. \tag{25}$$

Thus, we recover the spectrum and eigenfunctions of the type A Calogero system:

$$H_{\mathrm{Cal,A}}\Psi_n = E_{\mathrm{Cal,A}}(n)\Psi_n \qquad n\in\mathbb{Z}^N,\ n_1\geqslant n_2\geqslant\cdots\geqslant n_N\geqslant 0 \tag{26}$$

with

$$E_{\mathrm{Cal,A}}(\boldsymbol{n}) = \varepsilon_{0,\mathrm{A}} + 2(\omega_0 + \omega_0') \sum_{1 \leqslant j \leqslant N} n_j \qquad \Psi_{\boldsymbol{n}}(\boldsymbol{x}) = \Delta_{\mathrm{Cal,A}}^{1/2} p_{\boldsymbol{n},\mathrm{Cal,A}}(\boldsymbol{x}) \tag{27}$$

where $p_{\boldsymbol{n},\mathrm{Cal,A}}(\boldsymbol{x})$ denotes the symmetric polynomial determined by conditions A1 and A2, and with Δ_{A} (6) replaced by $\Delta_{\mathrm{Cal,A}}$ (25).

For type B, the transition to the Calogero system is very similar. The substitution $x_j \to \beta^{-1} x_j$ ($\partial_j \to \beta\partial_j$), $a \to g_0$, $b \to g_0' + 1/2$, $c \to (\beta^2\omega_0)^{-1}$, $d \to (\beta^2\omega_0')^{-1}$, and $H_B \to 4\beta^2\omega_0\omega_0' H_{\mathrm{B}}$ leads to a Hamiltonian of the form H_{B} (14) with $\exp(\pm \mathrm{i}\partial_j)$ replaced by $\exp(\pm\mathrm{i}\beta\partial_j)$ and

$$\begin{aligned} v(z) &= 1 + \beta g/(\mathrm{i}z) \\ w(z) &= \beta^{-2}\left(1 + \frac{\beta g_0}{\mathrm{i}z}\right)\left(1 + \frac{\beta g_0'}{(\mathrm{i}z + \beta/2)}\right)(1 + \mathrm{i}\beta\omega_0 z)(1 + \mathrm{i}\beta\omega_0' z). \end{aligned} \tag{28}$$

For $\beta \to 0$ one now obtains a differential operator of the form $H_{\mathrm{Cal,B}} - \varepsilon_{0,\mathrm{B}}$, where $H_{\mathrm{Cal,B}}$ is given by (2) with g_0 replaced by $g_0 + g_0'$ and ω_0 by $\omega_0 + \omega_0'$, and the constant $\varepsilon_{0,\mathrm{B}}$ reads

$$\varepsilon_{0,\mathrm{B}} = (\omega_0 + \omega_0') N (1 + 2(N-1)g + 2(g_0 + g_0')). \tag{29}$$

In the limit (and after dividing by a divergent numerical factor) the type B weight function becomes

$$\Delta_{\mathrm{Cal,B}} = \prod_{1 \leqslant j \neq k \leqslant N} |x_j + x_k|^g |x_j - x_k|^g \prod_{1 \leqslant j \leqslant N} |x_j|^{2(g_0 + g_0')} \mathrm{e}^{-(\omega_0 + \omega_0') x_j^2} \tag{30}$$

and the eigenfunctions become

$$H_{\mathrm{Cal,B}} \Psi_{\boldsymbol{n}} = E_{\mathrm{Cal,B}}(\boldsymbol{n}) \Psi_{\boldsymbol{n}} \qquad \boldsymbol{n} \in \mathbb{Z}^N,\ n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0 \tag{31}$$

with

$$E_{\mathrm{Cal,B}}(\boldsymbol{n}) = \varepsilon_{0,\mathrm{B}} + 4(\omega_0 + \omega_0') \sum_{1 \leqslant j \leqslant N} n_j \qquad \Psi_{\boldsymbol{n}}(\boldsymbol{x}) = \Delta_{\mathrm{Cal,B}}^{1/2} p_{\boldsymbol{n},\mathrm{Cal,B}}(\boldsymbol{x}^2) \tag{32}$$

where $p_{\boldsymbol{n},\mathrm{Cal,B}}(\boldsymbol{x}^2)$ denotes the even symmetric polynomial determined by conditions B1 and B2, and with Δ_{B} (20) replaced by $\Delta_{\mathrm{Cal,B}}$ (30).

Let us conclude by remarking that both our multivariable continuous Hahn and Wilson polynomials are limiting cases of a multivariable version of the Askey–Wilson polynomials [11] introduced by Koornwinder [12] as a generalization of Macdonald's polynomials associated with the root system BC_N [13, 14]. Koornwinder's multivariable Askey–Wilson polynomials are joint eigenfunctions of an algebra of commuting difference operators with trigonometric coefficients [15]. This algebra constitutes a complete set of quantum integrals for a difference version of the trigonometric BC_N-type Calogero–Sutherland system. Similar algebras consisting of commuting difference operators that are simultaneously diagonalized by our multivariable continuous Hahn and Wilson polynomials can be obtained as rational degenerations. The difference Hamiltonians considered in this letter are the simplest (i.e. lowest order) non-trivial operators in these algebras.

References

[1] Calogero F 1971 *J. Math. Phys.* **12** 419–36
[2] Brink L, Hansson T H and Vasiliev M A 1992 *Phys. Lett.* **286B** 109–11
[3] Olshanetsky M A and Perelomov A M 1983 *Phys. Reps.* **94** 313–404
[4] de Jeu M F E 1993 *Invent. Math.* **113** 147–62
[5] Ruijsenaars S N M 1987 *Commun. Math. Phys.* **110** 191–213

[6] van Diejen J F 1995 *J. Math. Phys.* **36** 1299–1323
[7] Atakishiyev N M and Suslov S K 1985 *J. Phys. A: Math. Gen.* **18** 1583–96
[8] Askey R 1985 *J. Phys. A: Math. Gen.* **18** L1017–9
[9] Wilson J A 1980 *SIAM J. Math. Anal.* **11** 690–701
[10] Askey R and Wilson J A 1982 *SIAM J. Math. Anal.* **13** 651–5
[11] Askey R and Wilson J A 1985 *Mem. Am. Math. Soc.* **54**
[12] Koornwinder T H 1992 *Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications (Contemporary Mathematics 138)* ed D St P Richards (Providence, RI: American Mathematical Society) pp 189–204
[13] Macdonald I G 1990 *Orthogonal Polynomials: Theory and Practice (NATO ASI Series C 294)* ed P Nevai (Dordrecht: Kluwer) pp 311–8
[14] Macdonald I G 1988 Orthogonal polynomials associated with root systems *Preprint* University of London
[15] van Diejen J F 1995 *Compos. Math.* **95** 183–233